BMath Algebra-IV
End-semestral test-2019

Instructions : Total time-3 hours. **Solve any five** problems. You may use results done in the class, without proofs. If you wish to use any assignment/tutorial problem, you should include its solution in your answer.

1. Let L be a separable extension of a field k, of prime degree p. Let $\theta \in L$ be such that $L = k(\theta)$. Let $\theta_1, \cdots, \theta_p$ be the conjugates of θ in an algebraic closure of k, with $\theta = \theta_1$. Assume that $\theta_2 \in L$. Prove that L/k is a cyclic extension. (10)

2. Let L/k be a Galois extension with $Gal(L/k) = \{\sigma_1, \cdots, \sigma_n\}$. Let $\alpha \in L$ be such that $\sigma_1(\alpha), \cdots, \sigma_n(\alpha)$ are all distinct. Prove that $L = k(\alpha)$. (10)

3. Let L/k be a finite Galois extension with $G := Gal(L/k)$. Let G' be the commutator subgroup of G and let $K := L^{G'}$. Show that K/k is an abelian extension. Show that K contains any subfield $M \supset k$ of L such that M/k is an abelian extension. (5+5)

4. Let $L/\mathbb{Q}$ be a Galois extension of odd degree contained in $\mathbb{C}$. Prove that $L \subset \mathbb{R}$. (10)

5. Construct a Galois extension of $\mathbb{Q}$ with Galois group isomorphic to $\frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}}$. (10)

6. Find all subfields of degree 2 over $\mathbb{Q}$ of the field **(i)** $\mathbb{Q}(\zeta_8)$ **(ii)** $\mathbb{Q}(\zeta_5)$ **(iii)** $\mathbb{Q}(\zeta_7)$. (6+2+2)

7. Let ζ be a primitive 7th root of unity in $\mathbb{C}$. Find an explicit field extension $K/\mathbb{Q}$ such that $\mathbb{Q}(\zeta) \subset K$ and K admits a radical root tower over $\mathbb{Q}$ with each step cyclic of prime degree. (10)